Multilateral Bargaining with Collective Proposal Control*

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Abstract

I study a model of multilateral bargaining in which multiple proposers simultaneously make offers on several pies. I identify a novel source of inefficient delay unique to multilateral bargaining – free-riding among proposers combined with the variability of proposal power. I establish that there exist stationary equilibria with delay and characterize the equilibrium agreement sets. In the worst equilibrium agents agree if and only if the proposal power is sufficiently concentrated. I compare the efficiency consequences of different voting rules, showing that voting rules requiring approval by greater majorities lead to more delay in the worst equilibrium.

1 Introduction

Writing legislative proposals is oftentimes a collective endeavor. When several agents possess the means and the expertise, as well as the desire to develop certain parts of a draft legislative bill or a resolution, the outcome is legislative proposals sponsored by multiple representatives. Indeed, the incidence of co-sponsored draft bills is high in legislative bargaining and bargaining in international organizations (Fowler 2006, Baller 2017: 475).[1] Likewise, bargaining within the governing coalition in parliamentary democracies requires

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[1] To illustrate, almost half of the bills introduced in the US Congress in 1973-2004 were cosponsored (Fowler 2006). Similarly, 38 percent of the bills introduced in the Chamber of Deputies in Argentina in 1983-2002 (Aleman et al. 2009: 98-99), and most bills introduced in the Chilean Chamber of Deputies in 1990-2002 (Crisp, Kanthak, and Leijonhufvud 2004: 710) were cosponsored.

reaching agreement on policies developed by multiple ministers who make proposals on the different portfolios that they control.

The budget approval process in the US Congress is a particularly persuasive example of bargaining with collective proposal control. There, the twelve Appropriations Subcommittees write the appropriation bills, so multiple agents have proposal power over different issues at the same time. The appropriation bills are then packaged together into one omnibus spending bill and voted on as a package.

The aim of the present paper is to explore the consequences of collective proposal control for the dynamics of the bargaining process. I start by introducing a model of multilateral bargaining with collective proposal control. A surplus is to be divided among the agents. The surplus consists of several parts, which I refer to as issues, or pies. In each period, proposers are selected randomly. An agent can be a proposer on a subset of the pies. Each proposer makes offers on the subset of the pies that she controls, and all agents accept or reject the offers on the table. Acceptance of at least q agents is needed for agreement.

My main finding is that the model admits stationary equilibria that are inefficient because they feature a delay in agreement. My other major results include a characterization of the worst equilibrium,[2] a comparison of the efficiency properties of different voting rules and a characterization of the set of all equilibrium agreement sets.

The main reason for the delay is free-riding among proposers: each proposer would like the other proposers to share surplus with the non-proposers to secure their support. Delay occurs because, given the conjecture that the other proposers are offering zero to all other agents under a particular realization of the proposal power, the equilibrium value can be large enough that a proposer is unwilling to make the offers leading to agreement all by herself. Thus the reason for the delay is twofold: it is the free-riding problem facilitated by the simultaneity of the proposals and the variability of the proposal power over time that enable the proposers to forego agreement today in hopes of reaching a better deal tomorrow. Importantly, a one-period version of the model I consider does not admit inefficient equilibria – the prospect of controlling more pies in the future is crucial for generating the inefficient delay.

My main contribution is identifying a new source of inefficiency specific to multilateral bargaining – free-riding among agents who have proposal power over certain issues. My model thus explains why reaching agreement is particularly difficult in multilateral, as compared to bilateral, bargaining. Because in a bilateral setting my model yields immediate agreement in the unique equilibrium, the reasons for delay isolated by my model are unique to the multilateral setting. This is in contrast to most of the existing models producing delay in

[2] The worst (symmetric) equilibrium is the equilibrium that yields the lowest payoffs to the agents.

bargaining, which feature delay in both bilateral and multilateral environments. A handful of papers with delay in multilateral and not bilateral settings obtain delay for very different reasons. In Ali (2006), for example, the reason for delay is persistent optimism in a game that is sufficiently long, while in Jehiel and Moldovanu (1995) and Cai (2000), who consider a seller engaged in a sequence of bilateral negotiations with buyers, the reasons are externalities and the possibility of partial agreements.

2 Delay: An Example

The first main result of the paper is that in my model there exist parameters such that there are stationary equilibria with delay. Here I provide a simple example to illustrate the result.

Suppose that there are three pies to divide, four agents and approval of all four agents is needed for agreement. With probability $\frac{1}{2}$, one of the players is in charge of all three pies, and with probability $\frac{1}{2}$, one player controls two pies and one player controls one pie.

Clearly, if the game lasts for only one period, there is a unique equilibrium with immediate agreement no matter who is in charge of the pies. In this equilibrium, the proposer on k pies keeps all k pies to herself and gives 0 to the non-proposers.

Next consider a game that lasts for two periods. When there is only one period remaining, the unique equilibrium is as described above and the expected payoff of an agent is $V = \frac{3}{4}$. We conjecture an equilibrium in which when there are two periods remaining agreement occurs if and only if one player is in charge of all three pies. To support disagreement when two players are proposers, we need that a proposer on two pies prefers not to make acceptable offers to the non-proposers provided that the other proposer is making zero offers. The payoff to making acceptable offers is $2 - 2\delta V$, while the payoff to disagreeing is δV. Thus we require that $2 - 2\delta V < \delta V$, which is equivalent to $\delta V_0 > \frac{2}{3}$. Because $V = \frac{3}{4}$, this is $\delta > \frac{8}{9}$.

Finally, consider an infinite-horizon game. Note first that there exists an equilibrium with no delay. In this equilibrium, when there are two proposers, the proposer on two pies offers $\frac{5}{6}\delta V$ to each non-proposer, while the proposer on one pie offers $\frac{1}{6}\delta V$ to each non-proposer. This yields payoffs $2 - 2\frac{5}{6}\delta V$ and $1 - 2\frac{1}{6}\delta V$ to the respective proposers. To ensure agreement, we need $2 - 2\frac{5}{6} > \delta V$ and $1 - 2\frac{1}{6}\delta V > \delta V$. Both of these conditions simplify to $\delta V < \frac{3}{4}$, which is satisfied for all δ since $V = \frac{3}{4}$ in the equilibrium with no delay.

We now conjecture an equilibrium in which agreement occurs if and only if one player is in charge of all three pies. Because agents are symmetric, the probability that one player is in charge of all three pies and agent i is a proposer is $\frac{1}{8}$. The value of an agent in this equilibrium satisfies

$$V = \frac{1}{8}(3 - 3\delta V) + \left(1 - \frac{1}{8}\right)\delta V$$

With probability $\frac{1}{8}$, agent i is a proposer on all 3 pies. Because there is agreement in this situation and there are three non-proposers, agent i makes offers δV equal to the discounted equilibrium values to each of the three non-proposers, which yields a payoff of $3 - 3\delta V$ to agent i. With probability $1 - \frac{1}{8}$, either one agent is in charge of all three pies but agent i is not the proposer or there are two proposers, in which case there is disagreement. In either case, agent i receives his discounted equilibrium value δV. Solving for V, we obtain $V = \frac{3}{4(2-\delta)}$.

In order to support disagreement when there are two proposers, we need that an agent who is a proposer on only two pies prefers not to make acceptable offers to the non-proposers provided that the other proposer is making zero offers. The payoff to making acceptable offers to the non-proposers is $2 - 2\delta V$, while the payoff to disagreeing is δV. Thus we require that $2 - 2\delta V < \delta V$, which is equivalent to $\delta V > \frac{2}{3}$. Thus an equilibrium with delay exists if the value in this equilibrium is sufficiently high. Using the formula for V, we can write this as $\delta > \frac{16}{17}$. We see that an equilibrium with delay exists if players are sufficiently patient.

The delay disappears as $\delta = e^{-r\Delta}$ goes to 1 (here Δ is the length of the time period and r is the discount factor). To see why, note that in the equilibrium with delay there is agreement if and only if one player is in charge of all three pies. Because the probability of this is $\frac{1}{2}$ in each period, for any fixed interval of time, as the length of the period Δ goes to 0, the probability that one player is in charge of all three pies at some point in this interval goes to one.

The intuition for the existence of an equilibrium with delay is as follows. Even though the size of the surplus is constant in every period and it is feasible to offer to each agent her discounted value in order to induce the agent to agree, because the proposal power is dispersed, cooperation among proposers is needed to make the agreement happen. If, for instance, a proposer conjectures that the other proposer is offering to the non-proposers half of the discounted value, then the proposer strictly prefers to also offer to the non-proposers half of the discounted value. If, on the other hand, a proposer conjectures that the other proposer is making zero offers, then in order to induce agreement the proposer would need to offer the whole discounted value to the non-proposers. Because the proposer controls only half of the available pies, the amount left to the proposer after making the acceptable offer to the non-proposers may be sufficiently small to make the proposer unwilling to make the offer. The proposer finds the burden of making the acceptable offer too heavy if the discounted value that she needs to offer is large enough, which explains why we needed that the equilibrium value be sufficiently large for the equilibrium with delay to exist.

3 Model

3.1 Description of the Model

There are n agents and m issues, which I also refer to as pies. Each issue generates one unit of surplus, so that m is the total surplus at stake in the bargaining game. We have $n > m$, so that the number of agents exceeds the number of pies.

Let us number the pies from 1 to m. A proposer recognition process is an element of $\Delta(\{1,\dots,n\}^m)$, where a realized vector specifies the identity of the proposer on each pie. The recognition process is iid over periods. I make a symmetry assumption that consists of two parts. First, for each vector of proposer identities $\psi \in \{1,\dots,n\}^m$, every permutation of ψ has the same probability. Thus, for example, if there are two pies, then proposer identities $\{1,2\}$ have the same probability as $\{2,1\}$. Second, for all j, $j' \in \{1,\dots,n\}$, if ψ, $\psi' \in \{1,\dots,n\}^m$ are such that $\psi_i \neq j'$ for any i, $\psi'_i = \psi_i$ for all i such that $\psi_i \neq j$ and $\psi'_i = j'$ for all i such that $\psi_i = j$, then ψ and ψ' have the same probability. For example, proposer identities $\{1,2\}$ have the same probability as $\{1,3\}$.

Before proceeding further, we find it useful to define a *multiset*. A multiset is a 2-tuple (A,z) where A is a set and $z : A \to \mathbb{N}$ is a function giving the number of occurrences of the element k in the multiset as the number $z(k)$.[3] To ease notation, we will write a multiset by listing all its elements. A *proposing partition* is a multiset Q such that for all $k \in Q$ we have $k \in \{1,\dots,m\}$ and $\sum_{k\in Q} k = m$.

To understand the definition of a proposing partition, consider an example. Suppose that there are two pies and three agents. Then the set of all proposing partitions is $\{\{2\},\{1,1\}\}$. If the proposing partition is $\{2\}$, then there is exactly one agent who is a proposer on both pies, and the two other agents are non-proposers. If the proposing partition is $\{1,1\}$, then there is an agent who is a proposer on exactly one pie, there is another agent who is a proposer on the other pie, and there is one agent who is a non-proposer. Observe that a proposing partition tells us whether one agent controls both pies or each pie is controlled by a different agent but it does not tell us the identities of the proposers.

Due to our symmetry assumption, we can describe the recognition process by the

[3] That is, a multiset is an unordered collection of objects where every element occurs a finite number of times, while a set is an unordered collection of *distinct* objects. For instance, $\{1,1\}$ is a multiset but not a set.

proposing partitions and their associated probability mass functions. In every period, proposing partitions are chosen according to a distribution on the space of all possible proposing partitions with a probability mass function f. The symmetry assumption implies that, given any realized proposing partition Q, for every number of pies $k \in Q$ and for every pair of agents i, i', the probability that agent i is a proposer on k pies under Q is equal to the probability that agent i' is a proposer on k pies under Q. I let $\mathcal{Q} = \operatorname{supp} f$ denote the set of proposing partitions that have a strictly positive probability under f.

I now describe the players' actions. Letting $\Delta_k = \big\{(x_1, \dots, x_n) \in \mathbb{R}^n : x_i \in [0, k] \text{ for all } i \in \{1, \dots, n\} \text{ and } \sum_{i=1}^n x_i = k\big\}$, a proposer on k pies chooses an offer $x \in \Delta_k$. The interpretation is that the proposer offers to give the amount x_i of her pies to each agent i. Given the offers of all proposers, each player i chooses $r_i \in [0, 1]$, with the interpretation that agent i accepts all offers with probability r_i and rejects all offers with probability $1 - r_i$. Note that I find it convenient to allow all agents, including the proposers, a choice of accepting or rejecting the offers.

The timing is as follows. First each proposer simultaneously makes proposals to all other agents on the pies that she controls. Then all agents accept or reject all the proposals. If $q \le n$ agents agree to the proposals, then the game ends and the agents receive their payoffs. In the event that the offers are accepted, the proposers get to keep the amount of the pies they control that they have not offered to anyone. If fewer than q agents agree, then no agreement is reached (on any pies) and in the next period proposers are chosen randomly again. I assume that $q > m$, that is, the number of agents whose support is needed for an agreement exceeds the number of pies.[4] The agents have a common discount factor $\delta \in (0, 1)$.

An outcome in period t is $\{\nu^t, X^t, R^t\}$: a proposer allocation function $\nu^t(i) : \{1, \dots, n\} \to \{0, 1, \dots, m\}$ such that $\{\nu^t(i) > 0 : i \in \{1, \dots, n\}\} = Q^t$ (where Q^t is the realized proposing partition in period t)[5] indicating how many pies each agent i was a proposer on, a collection of offers of the proposers $X^t \in \times_{i \in \{1,\dots,n\} : \nu^t(i) > 0} \Delta_{\nu^t(i)}$ and a collection of the votes of each agent to accept or reject $R^t \in \{0, 1\}^n$. A T-period history then is $h^T = \{\nu^t, X^t, R^t\}_{t \le T}$.

Let H^t denote the space of all t-period histories, and let $\mathcal{V}$ denote the space of proposer allocation functions. Given $\nu^t \in \mathcal{V}$, a strategy at the proposal stage in period t for agent i who is a proposer on k pies is $\tilde{\chi}_{i,k,\nu}(h^{t-1}) : H^{t-1} \to \Delta_k$. A strategy at the response stage in

[4] A sufficient condition for this assumption to be satisfied is that $n > 2m$ and $q > \frac{n}{2}$, that is, it is sufficient that the number of agents is sufficiently large relative to the number of pies and the voting rule requires the support of at least a simple majority of the voters.

[5] There is a slight abuse of notation here because $\{\nu^t(i) > 0 : i \in \{1, \dots, n\}\}$ is an ordered set, while Q^t is an unordered set. What we mean is that there is a way of ordering Q^t such that it is equal to $\{\nu^t(i) > 0 : i \in \{1, \dots, n\}\}$.

period t for agent i is $\tilde{r}_{i,\nu}(h^{t-1}, X^t) : H^{t-1} \times \times_{i \in \{1,\dots,n\}:\nu^t(i)>0} \Delta_{\nu^t(i)} \to [0,1]$.

I assume that the proposers have access to a public randomization device. The role of the public randomization device is to ensure that a (symmetric) pure strategy equilibrium exists when the voting rule is not unanimity. In this setting, the proposers have to coordinate on the identities of the members of the minimal non-proposing winning coalition, and a public randomization device aids the coordination.

3.2 Equilibrium Definition

I focus on pure strategy symmetric subgame perfect equilibria that are stationary (i.e. the offers depend only on the proposing partition Q in the current period and the acceptance decisions depend only on the current offers and the current Q).

If, given Q, there are fewer than q proposers, a minimal non-proposing winning coalition is a subset of the non-proposers such that the number of the proposers plus the number of the members of this subset is q. If there are at least q proposers, a minimal non-proposing winning coalition is an empty set. I let $\mathcal{M}_Q$ denote the set of all minimal non-proposing winning coalitions given Q.

Our equilibrium concept requires that a proposer makes the same offer to all other proposers, and makes the same offer to all non-proposers in a minimal non-proposing winning coalition selected by the public randomization device. Thus, letting $\Delta_k^Q = \{(y, y') : y, y' \in [0, k], (|Q| - 1)y + (q - |Q|)y' \leq k\}$, a proposer on k pies chooses an offer in Δ_k^Q when the proposing partition is Q.

We restrict our attention to strategies of the following form. Given $Q \in \mathcal{Q}$, a strategy at the proposal stage for a proposer on k pies is a mapping $\chi_{k,Q}(M) : \mathcal{M}_Q \to \Delta_k^Q$ from the space of minimal non-proposing winning coalitions to the space of feasible offers Δ_k^Q. Given $Q \in \mathcal{Q}$, a strategy for an agent at the response stage is a mapping $r_Q(x) : \times_{k \in Q} \Delta_k^Q \to [0, 1]$ from the space of feasible offer profiles to $[0, 1]$.

An equilibrium is then a collection indexed by $Q \in \mathcal{Q}$ consisting of a public randomization device that generates a uniform distribution over $\mathcal{M}_Q$, a collection of mappings $\chi_{k,Q}(M) : \mathcal{M}_Q \to \Delta_k^Q$ for each proposer on k pies for each $k \in Q$, and a collection of mappings $r_Q(x) : \times_{k \in Q} \Delta_k^Q \to [0, 1]$ for each agent.

For convenience, I impose an innocuous genericity assumption: whenever a proposer weakly prefers to make strictly positive offers rather than zero offers, the proposer strictly prefers to make these offers.[6] [7]

I focus on the equilibria in which only non-proposers receive strictly positive offers. Lemma 1 in the Appendix shows that it is without loss of generality: for any equilibrium in which some proposers receive strictly positive offers there exists an equilibrium in which only non-proposers receive strictly positive offers and the agreement set is the same.

I require that agents use stage-undominated voting strategies (Baron and Kalai 1993).[8] It can be shown that this implies that the equilibria that I construct are stage undominated. I also require that proposers do not make (net) offers that leave them strictly less than their discounted equilibrium continuation values.[9]

An *agreement set* for the equilibrium σ is a set of proposing partitions $\mathcal{Q}(\sigma)$ that lead to agreement. I focus on the equilibria in which proposers make zero offers under the proposing partitions $Q \notin \mathcal{Q}(\sigma)$. Lemma 2 in the Appendix shows that this is, in a sense, without loss of generality: for any equilibrium in which some proposer makes strictly positive offers under a proposing partition $Q \notin \mathcal{Q}(\sigma)$ there exists an equilibrium σ' with the same agreement set in which all proposers make zero offers under the proposing partitions $Q \notin \mathcal{Q}(\sigma')$.

4 Delay and Immediate Agreement

4.1 Measure of Resources for Overcoming Gridlock

In order to achieve agreement, proposers have to share some of the surplus with non-proposers. This leads to a free-riding problem among proposers: each would like the other proposers to share surplus with the supporters while keeping her own surplus. The proposing partitions Q differ in the extent to which they lead to free-riding problems.

I let $s_Q = q - |Q|$ denote the number of non-proposers whose support is needed for agreement under Q provided that all proposers support the agreement. I refer to s_Q as the size of the minimal non-proposing winning coalition, or supporters.[10]

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[6] A proposer making zero offers means that a proposer is offering to give zero to all other agents and is keeping all the pies that she controls for herself.

[7] I state the assumption informally for the ease of exposition. The assumption can be stated formally as requiring that certain inequalities hold strictly. The assumption is satisfied for generic values of δ.

[8] See section A.1 in the Appendix for more details.

[9] Given that agents use stage-undominated voting strategies, such offers must be rejected at the voting stage. The requirement rules out unreasonable equilibria in which proposers vote against their own offers.

[10] Observe that our assumption that $q > m$ implies that $q > |Q|$ for all Q. If instead we had $q < |Q|$, that is, if the number of agents whose support was needed for an agreement was smaller than the number of the proposers, then $s_Q = 0$. It can be shown that in this case in any equilibrium there must be agreement under such a proposing partition and that all the proposers must vote in favor of it, implying that agreements

I define $D_Q = \max\{k \in Q\}$ as the maximum number of pies an agent is a proposer on given that the proposing partition is Q. Then R_Q denotes the measure of resources available for overcoming gridlock at Q:

$$R_Q := \frac{D_Q}{s_Q + 1}$$

We can interpret R_Q as concentration of the proposal power.

4.2 The Value Function

I first show that the value of an agent in a symmetric equilibrium can be written as a function only of the probability of the proposing partitions under which agents agree. Suppose that in equilibrium there is agreement under partitions $\mathcal{P}$. Then agreement happens with probability $\sum_{Q\in\mathcal{P}} f(Q)$, and we can compute the value function in this equilibrium as

$$\begin{aligned} U(\mathcal{P}, f) &= \frac{m}{n}\sum_{Q\in\mathcal{P}} f(Q) + \delta\left(1 - \sum_{Q\in\mathcal{P}} f(Q)\right) U(\mathcal{P}, f) \\ U(\mathcal{P}, f) &= \frac{\frac{m}{n}\sum_{Q\in\mathcal{P}} f(Q)}{1 - \delta + \delta\sum_{Q\in\mathcal{P}} f(Q)} \end{aligned}$$

For characterizing the worst equilibrium, cutoff strategies are important. A *cutoff strategy* is one where agreement sets have the form

$$\mathcal{Q}(\sigma) = \{Q \in \mathcal{Q} : R_Q \geq b\}$$

Given the PMF f of the distribution of the proposal power, I define a CDF F with respect to the order on the space of the proposing partitions induced by R_Q[11] as $F(Q) = \sum_{Q':R_{Q'}\leq R_Q} f(Q')$. Moreover, given Q such that there exists $Q' \in \mathcal{Q}$ with $R_{Q'} < R_Q$, I let

$$Q_- = \{Q' \in \mathcal{Q} : R_{Q'} < R_Q, R_{Q'} \geq R_{Q''}\ \forall\ Q'' \in \mathcal{Q} : R_{Q''} < R_Q\}$$

denote the proposing partition that immediately precedes Q according to the order induced by R_Q. If there does not exist $Q' \in \mathcal{Q}$ with $R_{Q'} < R_Q$, I set $R_{Q_-} = 0$.

To compute the value function in a cutoff equilibrium profile, suppose that agreement

would involve non-minimal winning coalitions.

[11] For simplicity, I assume that the parameters are such that the order on the space of the proposing partitions induced by R_Q is strict.

occurs under all Q such that $R_Q \geq b$ for some $b \neq R_Q$ for any $Q \in \mathcal{Q}$.[12] Observe that $1-F(b)$ is the probability that Nature chooses Q with gridlock resources $R_Q > b$. Agreement thus occurs with probability $1 - F(b)$. The value function is then given by

$$\begin{aligned} V &= \frac{m}{n}(1 - F(b)) + \delta F(b) V \\ V(b) &= \frac{\frac{m}{n}(1 - F(b))}{1 - \delta(F(b))} \end{aligned}$$

Observe that V is a non-increasing step function: the larger the cutoff gridlock resources R_Q, the smaller the agreement set and the probability of agreement, and thus the lower the payoffs.

4.3 Delay

I start by making an assumption on the parameters that allows for the largest possible set of pure strategy equilibria to exist.[13] First, I assume that there exist $Q \in \mathcal{Q}$ such that $R_Q \geq \delta V(R_Q)$. Next, I let

$$Q^* = \{Q \in \mathcal{Q} : R_Q \geq \delta V(R_Q), R_Q \leq R_{Q'} \ \forall \ Q' \in \mathcal{Q} : R_{Q'} \geq \delta V(R_{Q'})\}$$

denote the lowest Q (according to the order induced by R_Q) satisfying $R_Q \geq \delta V(R_Q)$.[14] Finally, I assume that[15]

$$R_{Q^*_-} < \delta V(R_{Q^*})$$

The assumption is satisfied if proposing partitions with high gridlock resources R_Q are sufficiently likely. For example, making the proposing partition $\{m\}$ sufficiently likely would ensure that the condition is satisfied. Moreover, higher per-agent surplus $\frac{m}{n}$ and discount factor δ make the assumption more likely to hold.

Fixing a distribution of the proposal power, an equilibrium is said to be the worst if

[12] It is straightforward to extend V to values b such that $b = R_Q$ for some $Q \in \mathcal{Q}$: set $V(R_Q) = \frac{\frac{m}{n}\left(1-F\left(R_{Q_-}\right)\right)}{1-\delta F\left(R_{Q_-}\right)}$.

[13] If this assumption is violated, the value function changing discontinuously when a proposing partition is added to an agreement set may lead to non-existence of some of the pure strategy equilibria described below. Allowing for mixed strategies would restore equilibrium existence at the cost of making the exposition more cumbersome. In particular, if mixed strategies are allowed, the statement in Theorem 1 that there exists an equilibrium with delayed agreement if and only if $\min_{Q \in \mathcal{Q}} R_Q < \delta \frac{m}{n}$ is true without any additional parameter assumptions.

[14] Note that, because there exist $Q \in \mathcal{Q}$ such that $R_Q \geq \delta V(R_Q)$, Q^* is well-defined.

[15] Note that V is a function that depends on $F(R_{Q^*_-})$, $\frac{m}{n}$ and δ and is *not* an equilibrium object.

no other equilibrium that exists given this distribution yields strictly lower payoffs to the agents.[16] I now present a result showing when delay occurs and how much delay is possible.

Theorem 1.

1. There always exists an equilibrium with immediate agreement.
2. There exists an equilibrium with delayed agreement if and only if $\min_{Q \in \mathcal{Q}} R_Q < \delta \frac{m}{n}$. If there exists an equilibrium with delay, then the worst equilibrium involves agreement under proposing partitions $\mathcal{C}_f$ that satisfy

$$\mathcal{C}_f = \{Q \in \mathcal{Q} : R_Q \geq \delta V(R_Q)\}$$

The first part of the theorem says that there is always an equilibrium with no delay. The second part says that there is an equilibrium with delay if and only if a condition on the primitives is satisfied. The condition holds if the discount factor and the per-agent surplus are large enough.

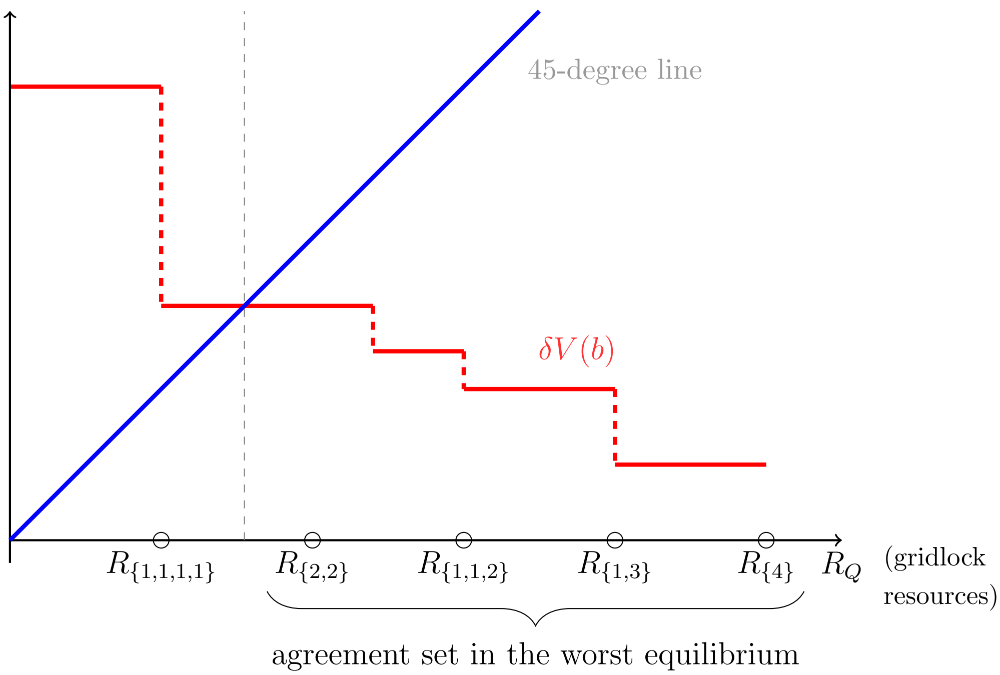
The second part of the theorem also says that a collection of proposing partitions is an agreement set in the worst equilibrium if and only if it is the set of all partitions $Q \in \mathcal{Q}$ such that the gridlock resources R_Q exceed $\delta V(R_Q)$, the discounted value function in a cutoff equilibrium profile. That is, the agreement set is a set of the partitions under which the proposal power is sufficiently concentrated. Henceforth, I use $\mathcal{C}_f$ to denote the agreement set in the worst equilibrium given that the distribution of the proposal power is f.

To understand the result, two observations are important. First, observe that delay happens in the model because the burden of making acceptable offers is shifted to certain proposers only. The incentive to not make acceptable offers (which leads to delay) is the greatest if there is only one individual who is tasked with making all the offers. Therefore, the worst equilibrium is one in which, whenever equilibrium prescribes disagreement under a proposing partition, each proposer believes that the other proposers are making zero offers.

Second, observe that, because the amount left to the proposer is increasing in the number of pies she controls, if the proposer on the maximum number of pies is unwilling to make acceptable offers, then neither are proposers on a smaller number of pies. Moreover, because the amount left to the proposer is decreasing in the number of the non-proposers she has to buy off, if a proposer on a given number of pies is not willing to make acceptable offers to the non-proposers under some proposing partition, then, keeping the number of pies a proposer controls fixed, she is not willing to make acceptable offers to a greater number of

[16] Recall that we focus on symmetric equilibria, which implies that all agents obtain the same payoffs. The reason agents' payoffs may differ across equilibria is that the amount of delay in these equilibria may differ.

Figure 1: Worst Equilibrium as An Intersection



non-proposers.

These two observations imply the following. Suppose that in the worst equilibrium there is disagreement under some proposing partition Q. Then it must be the case that the proposer on the maximum number of pies prefers to make zero offers under the conjecture that all other proposers make zero offers. We claim that this implies that there must be disagreement under all proposing partitions Q' satisfying $R_{Q'} < R_Q$. The reason is that, because gridlock resources are smaller under Q', the proposer either controls fewer pies or needs to buy off more non-proposers. Thus if she was unwilling to do so given the worst conjecture about the behavior of the other proposers under Q, she must be unwilling to do so given the worst conjecture under Q'.

Figure 1 illustrates the geometric interpretation of the worst equilibrium as an intersection. The figure depicts the gridlock resources for the proposing partitions $\{1,1,1,1\}$, $\{2,2\}$, $\{1,1,2\}$, $\{1,3\}$ and $\{4\}$ on the x-axis. For this example, I assume that $n = 6$, $\delta = \frac{9}{10}$ and the voting rule is unanimity. The discounted value function in a cutoff equilibrium profile δV is drawn in red, and the 45-degree line is drawn in blue. The figure shows that $\mathcal{C}_f$, the agreement set in the worst equilibrium, consists of the proposing partitions $\{2,2\}$, $\{1,1,2\}$, $\{1,3\}$ and $\{4\}$, which are precisely the partitions with R_Q above $\delta V(R_Q)$.

We now characterize all equilibrium agreement sets. Before stating the main result, I define the set $\mathcal{G}$.

Definition 1.

$$\mathcal{G} = \left\{ Q \in \mathcal{Q} : R_Q \geq \delta \frac{m}{n} \right\}$$

Proposition 1 characterizes the set of all the equilibrium agreement sets.

Proposition 1.

1. *(Increasing Mass on Agreement Set) If a collection of proposing partitions is an agreement set in some equilibrium, then it remains an equilibrium agreement set if we increase the probability of this collection.*
2. *(Guaranteed agreement set) In every equilibrium, there is agreement under any Q with $R_Q > \delta \frac{m}{n}$. There is always agreement when one agent is a proposer on all pies.*
3. *(Worst Agreement Set Expands) Any superset of $\mathcal{C}_f$ is an agreement set in some equilibrium that exists under f.*
4. *If $\sum_{Q \notin \mathcal{C}_f} f(Q) < f(Q')$ for all $Q' \in \mathcal{C}_f \setminus \mathcal{G}$, then the set of all equilibrium agreement sets under f is equal to the set of all supersets of $\mathcal{C}_f$.*

The intuition is as follows. It can be shown that agents disagree under a proposing partition Q if the equilibrium value is sufficiently large relative to the gridlock resources R_Q. The reason is that a larger equilibrium value means that, in order to achieve agreement, a proposer needs to give more to the non-proposers and that the value the proposer receives in the event of disagreement is larger. Increasing the probability of the proposing partitions in the agreement set or adding partitions to the agreement set increases the equilibrium payoff and thus preserves incentives for disagreement under the partitions where agents are supposed to disagree.

4.4 Finite Horizon Equilibria

Given a game that lasts for T periods, I let $\mathcal{Q}_t(\sigma)$ denote the agreement set in an equilibrium σ given that t periods have passed. Proposition 2 provides a result relating equilibria in finite-horizon games to equilibria in the game with infinite horizon.

Proposition 2. *If $\min_{Q \in \mathcal{Q}} R_Q > \delta \frac{m}{n}$, then in any game (with finite or infinite horizon) there is immediate agreement. If $\min_{Q \in \mathcal{Q}} R_Q \leq \delta \frac{m}{n}$, then for any equilibrium σ of the infinite-horizon game there exists an equilibrium σ^T of a T-period game such that $\mathcal{Q}_t\left(\sigma^T\right) = \mathcal{Q}(\sigma)$ for all $t < T - 1$.*

Proposition 2 shows that for any equilibrium in the infinite-horizon game there exists a sequence of equilibria of finite-horizon games that converges to the infinite-horizon equilibrium as the game grows long. In fact, a stronger result is true: for any equilibrium in the infinite-horizon game and any number of periods T there is an equilibrium in the T-period game that coincides exactly with the equilibrium in the infinite-horizon game in all periods except possibly in the last period T.

The intuition for the result relies on the fact that agents disagree if the equilibrium value is large enough. In the last period T of the finite-horizon game, there must be agreement with probability one, which yields the largest possible continuation value at $T-1$. Because this value is larger than any value in the infinite-horizon equilibrium, at any proposing partition where we had disagreement in the infinite-horizon game, we can sustain disagreement in the game with a finite horizon.

5 The Worst Equilibrium

5.1 Simple Equilibrium

An equilibrium σ is said to be *simple* if whenever $Q \in \mathcal{Q}(\sigma)$ a proposer makes a strictly positive offer under Q if and only if she is a proposer on D_Q pies. Thus an equilibrium is simple if the only agents who make strictly positive offers are the most powerful ones. The following Proposition shows that there is a simple worst equilibrium.

Proposition 3. *There exists a worst equilibrium that is simple. Moreover, if for all proposing partitions that have a strictly positive probability there is exactly one proposer on the largest number of pies, then there exists a unique simple equilibrium.*

The first part of Proposition 3 asserts that there exists a worst equilibrium that is simple. To understand why this is the case, let b^* be the unique b such that $\delta V(b) \geq b \geq \delta V(b')$ for $b' \geq b^*$. Observe that b^* is precisely the point at which a proposer on the maximum number of pies D_Q prefers not to make acceptable offers for all Q with R_Q below b^* under the conjecture that all other proposers are making zero offers and would make acceptable offers for all Q with R_Q above b^*, also under the conjecture that all other proposers are making zero offers. In a simple symmetric equilibrium, if there are several agents who are proposers on D_Q pies, then they must all make strictly positive acceptable offers to the non-proposers. However, if the proposer is willing to make acceptable offers alone, then the proposer is certainly willing to make acceptable offers when the burden of making the offers is shared. Therefore, there is a simple worst equilibrium.

The second part of Proposition 3 says that if there is only one proposer on the maximum number of pies, then there is a unique simple equilibrium. In a simple equilibrium the proposer must prefer to make acceptable offers for Q in the agreement set and not to make acceptable offers for Q in the disagreement set under the conjecture that all other proposers are making zero offers. As explained above, under this conjecture the proposer on D_Q pies is willing to make acceptable offers if and only if Q is such that $R_Q \geq \delta V(R_Q)$, so the simple equilibrium is unique and coincides with the worst equilibrium.

5.2 Comparison of Voting Rules

In this section I compare the properties of different voting rules in the worst equilibrium.

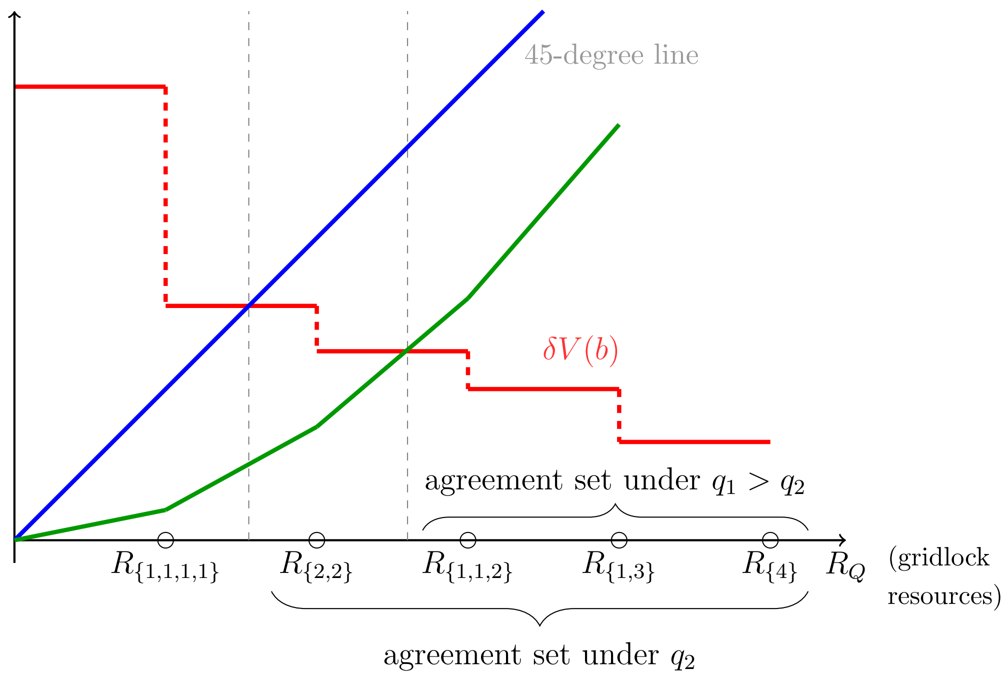
Proposition 4. *In the worst equilibrium delay is longer and payoff is lower under the voting rules that require a larger majority.*

The reason that smaller majority requirements q generate smaller delay is that, because under smaller q proposers need fewer supporters to get their proposals passed, they can keep a greater share of the pies that they control for themselves after making acceptable offers to the minimal non-proposing winning coalition. This makes it more difficult to sustain equilibria with delay in which a proposer is unwilling to make acceptable offers because the amount of pies left to the proposers after making the offers is too small compared to her discounted value in the future.

Figure 2 provides a geometric intuition for the result. As we increase the majority requirements, $R_Q = \frac{D_Q}{s_Q+1}$ decreases for all Q because more supporters s_Q need to be paid. On the other hand, $V(R_Q)$ does not change because the payoff in a symmetric equilibrium with a *given* agreement set does not depend on the voting rule. Then the gridlock resources that make the marginal most powerful proposer indifferent between making acceptable offers and not go up. Because the agreement set in the worst equilibrium is the set of all partitions Q with R_Q exceeding the gridlock resources that make the proposer indifferent, the agreement set in the worst equilibrium shrinks, which increases delay and lowers payoffs.

A comparison of the welfare properties of voting rules in my model with the welfare properties of voting rules in the models of bargaining with stochastic surplus due to Merlo and Wilson (1995) and Eraslan and Merlo (2002) is of interest. Eraslan and Merlo (2002) show that in a setting with one proposer and a stochastic surplus the payoffs under unanimity are higher than the payoffs under non-unanimous agreement rules because agents agree too soon under non-unanimous voting rules. That is, in their model, delay is efficient, and non-unanimity rules lead to an inefficiently small amount of delay. In contrast, in my model, because the size of the surplus is constant, all delay is inefficient, and rules requiring greater

Figure 2: Comparison of the Voting Rules



majorities are worse because they induce greater delay.

Thus my model yields the conclusions that are the opposite of those reached by Eraslan and Merlo (2002): while in their model the unanimity rule is better than non-unanimity rules, in my model non-unanimity rules lead to higher payoffs (in the worst equilibrium) than the unanimity rule.

6 Related Literature

The paper is related to several strands of literature on bargaining with complete information. Most notably, it is related to the Baron-Ferejohn model of legislative bargaining (Baron and Ferejohn 1989) in which one proposer on one pie is randomly selected in every period. This model has been extended in various directions (Eraslan 2002, Kalandrakis 2015, Eraslan and Merlo 2017). Unlike the Baron-Ferejohn model, the model in the present paper allows for multiple pies and multiple proposers who make the offers on the pies they control simultaneously. Whereas in the Baron-Ferejohn model there must be immediate agreement in any stationary equilibrium,[17] my model admits stationary equilibria with delay.

[17] Provided that agents use stage-undominated voting strategies (Baron and Kalai 1993).

The literature on bargaining with complete information has identified several reasons leading to delay in agreement: the size of the available surplus changing stochastically[18] (Merlo and Wilson 1995, Eraslan and Merlo 2002), non-common prior beliefs about recognition probabilities[19] (Yildiz 2003, Ali 2006), non-stationarity of the equilibrium (Fernandez and Glazer 1991, Haller and Holden 1990), and repetition of a stage game that has an inefficient Nash equilibrium (Dekel 1990, Chatterjee and Samuelson 1990). None of these features are present in my model – thus the reason for delay that I identify is novel.

Finally, the literature on multi-issue bargaining (Fershtman 1990, Inderst 2000, In and Serrano 2004) is related. The papers in this literature find that agreement is reached without delay in stationary equilibria and that the bargaining agenda can have distributive effects.

[18]Also related is a model by Cai (2000) in which one proposer engages in a sequence of bilateral negotiations with responders. If a responder accepts an offer, she receives a cash payment immediately and leaves the bargaining process. Cai (2000) shows that these endogenous changes in bargaining environment can produce delay.

[19]Ortner (2013) combines stochastic surplus with non-common prior beliefs about recognition probabilities.

Appendix

A.1 Stage Undominated Equilibria

It is important to point out that the equilibria with delay that I construct do not rely on the agents using weakly dominated strategies. In order to check whether the equilibria I characterize are weakly dominated or not, I need to first define the notion of weak dominance appropriate for my game. A definition of weakly undominated equilibria commonly used in bargaining games is that of stage undominated equilibria, introduced by Baron and Kalai (1993). A stage game starts with the proposers being drawn randomly and ends with an acceptance or a rejection of the offers of the proposers. An equilibrium is said to be stage undominated if the strategies it induces in every stage game are weakly undominated for any agent.[20] Letting V^σ denote the value of an agent in the equilibrium σ, a stage-undominated voting strategy is a voting strategy such that an agent accepts the offers made to her if and only if the sum of the offers x satisfies $x \geq \delta V^\sigma$.

It can be shown that, provided that agents use stage-undominated voting strategies, the equilibria that I construct are both weakly undominated and (generically) stage undominated. Informally, the reason is that there is no strategy for a proposer that is optimal no matter what the other agents do: if the other agents accept when the proposer makes zero offers, then the proposer strictly prefers to make zero offers, while if the other agents reject when the proposer makes zero offers, then the proposer may strictly prefer to make strictly positive offers.

B Proofs

Lemma 1 (No Transfers Among Proposers). *If there exists an equilibrium in which under some $Q \in \mathcal{Q}(\sigma)$ some proposer receives a strictly positive offer, then there exists an equilibrium in which no proposer receives strictly positive offers and the set of the proposing partitions under which there is agreement is the same.*

Proof of lemma 1.

Let V denote a value in some equilibrium. We first show that $\delta V < 1$. Suppose this was not the case. Then $\delta n V \geq n$. Because $n > m$, this implies that $nV > \delta n V > m$. But

[20] Recall that a strategy of an agent is weakly undominanted if there does not exist a strategy that weakly dominates it, that is, if there does not exist a strategy that yields to the agent a weakly higher payoff no matter what strategies the other agents use and a strictly higher payoff for some strategy profile of the other agents.

then the sum of the agents' values nV exceeds the value of the available pies m, which is a contradiction.

Fix an equilibrium and a proposing partition Q such that there is disagreement under Q in this equilibrium. Consider the strategy profile in which each proposer makes zero offers to all other agents and keeps all her pies for herself under Q. Then all non-proposers vote against the proposals, which implies that there is disagreement under Q, as required.

Next, fix a proposing partition Q such that there is agreement under Q in the equilibrium that we fixed. Let P denote the set of the agents who are proposers under Q. For each agent $i \in P$, let X_i denote the sum of the offers he makes to the other agents. For each agent i, let Y_i denote the sum of the offers he receives from the other agents.

We will define a new set of offers as follows. Let x_{ij} denote the new offer that agent i makes to agent j. Set $x_{ij} = 0$ for all $j \in P \setminus i$.

For each $i \in P$, let $Z_i = X_i - Y_i$. Z_i is the net sum of the offers that proposer i makes to the other agents. We claim that we must have $Z_i \geq 0$.

Suppose this was not the case and we had $Z_i < 0$. Fix an agent j such that agent i receives a strictly positive offer x from agent j (note that the fact that $Z_i < 0$ implies that such an agent must exist). Consider agent j instead offering $x' < x$ to agent i (and keeping $x - x'$ for herself) and let the resulting net sum of the offers that proposer i makes to the other agents be denoted by Z_i'. Let $x' < x$ be chosen such that $Z_i' \leq 0$ (note that this is feasible because $Z_i < 0$).

Making the offer x' instead of x is a strictly profitable deviation for agent j if agent i still accepts the offer. Because $Z_i' \leq 0$, the payoff to agent i who is a proposer on $k \geq 1$ pies from accepting would be at least $k \geq 1$, while the payoff to rejecting would be δV. By the first paragraph of the proof, we have $1 > \delta V$, so that $k \geq 1 > \delta V$. Then agent i must accept this offer (because agents use weakly undominated voting strategies). This is a contradiction.

Define
$$\alpha_i = \frac{Z_i}{\sum_{j \in P} Z_j}$$

Note that the fact that there is agreement under Q implies that $Z_i > 0$ for some $i \in P$. Therefore, we have $\sum_{j \in P} Z_j > 0$, which implies that α_i is well-defined.

α_i is the ratio of the sum of the net offers proposer i makes to the other agents to the sum of the net offers all proposers make to the other agents. Observe that, because $Z_i \geq 0$ for all i, we have $\alpha_i \geq 0$ for all i.

Suppose that the public randomization device generates a uniform distribution over $\{J \subseteq N_Q : |J| = s_Q\}$. Given that the realization of the public randomization device is J, for each $i \in P$ and $k \in J$, let $x_{ik} = \alpha_i Y_k$. That is, we let each proposer i make an offer to a non-proposer k in the subset of the non-proposers J that gives to the non-proposer k in J a share α_i of the sum of his previous offers Y_k.

Note that for each $i \in P$, the net transfers with the original offers are $-Z_i = Y_i - X_i$. Observe also that we have $\sum_{j \in P} X_j = \sum_i Y_i$, that is, the sum of all offers made by the proposers to the other agents must be equal to the sum of the offers received by all agents. This implies that $\sum_{j \in P} Z_j = \sum_{k \in J} Y_k$, that is, the sum of the net offers made by the proposers to the other agents is equal to the sum of the offers received by the non-proposers in the subset J.

By construction, the sum of the offers received by each agent $i \in J$ is the same under the previous offers and under the new offers. Under the new offers, the net offers made by each agent $i \in P$ to the other agents are $-\sum_{k \in J} x_{ik} = -\alpha_i \sum_{k \in J} Y_k = -\frac{Z_i}{\sum_{j \in P} Z_j} \sum_{k \in J} Y_k = -Z_i$ because $\sum_{j \in P} Z_j = \sum_{k \in J} Y_k$. Therefore, the new set of offers gives the same payoffs to all agents. Thus it is consistent with equilibrium for the agents to make and accept these offers under Q. ■

Lemma 2. *Fix a distribution of proposal power f. Consider the class of pure strategy symmetric stationary subgame perfect equilibria in which agents use stage-undominated voting strategies and proposers do not make net offers that leave them strictly less than their discounted equilibrium continuation values. If $\mathcal{P} = \mathcal{Q}(\sigma')$ for some equilibrium σ' that exists given f such that some proposer makes strictly positive offers under $Q \notin \mathcal{Q}(\sigma')$, then (given f) there exists an equilibrium σ such that $\mathcal{P} = \mathcal{Q}(\sigma)$ and all proposers make zero offers under $Q \notin \mathcal{Q}(\sigma)$.*

Proof of lemma 2.

Fix $Q \notin \mathcal{Q}(\sigma')$ such that some proposer makes strictly positive offers when the proposing partition is Q. Let x denote the sum of offers that an agent in the minimal non-proposing winning coalition (MNWC) receives from the proposers under Q.

Note that the requirement that proposers do not make net offers that leave them strictly less than their discounted equilibrium continuation values, combined with the requirement that agents use stage-undominated voting strategies, implies that the proposers always vote in favor of the proposals. Then, because there is disagreement under Q in σ' and agents use stage-undominated voting strategies, we must have $x < \delta V^{\sigma'}$ for an agent in the MNWC (note that this inequality must be satisfied for all members of the MNWC because all members of this coalition are treated symmetrically by proposers).

Let us use y to denote the offer of a proposer on D_Q pies to a member of MNWC in equilibrium σ'. Then, in order to achieve agreement, a proposer on D_Q pies would have to give $\delta V^{\sigma'} - (x-y)$ (rather than y) to each of the s_Q of the members of the MNWC. By lemma 1, for the purposes of determining the payoffs that are attainable in equilibrium, it is without loss of generality to focus on the equilibria in which no proposer receives strictly positive offers. Then, because no proposer receives strictly positive offers, a proposer on D_Q pies keeps D_Q less the offers she makes to the non-proposers. Then disagreement under Q in σ' implies that we must have $D_Q - s_Q(\delta V^{\sigma'} - (x-y)) < \delta V^{\sigma'}$, that is, a proposer on D_Q pies is unwilling to make the offers leading to an agreement. Because $D_Q - s_Q \delta V^{\sigma'} \leq D_Q - s_Q(\delta V^{\sigma'} - (x-y))$, the fact that $D_Q - s_Q(\delta V^{\sigma'} - (x-y)) < \delta V^{\sigma'}$ implies that $D_Q - s_Q \delta V^{\sigma'} < \delta V^{\sigma'}$.

By lemma 4, in any equilibria σ and σ' satisfying $\mathcal{Q}(\sigma) = \mathcal{Q}(\sigma')$ we must have $V^{\sigma'} = V^{\sigma}$. Then, because $V^{\sigma'} = V^{\sigma}$, the fact that $D_Q - s_Q \delta V^{\sigma'} < \delta V^{\sigma'}$ implies that $D_Q - s_Q \delta V^{\sigma} < \delta V^{\sigma}$. Therefore, there must also be disagreement under Q in equilibrium σ (given the conjecture that all other proposers make zero offers).

Let the strategies in σ under proposing partitions $Q \in \mathcal{Q}(\sigma)$ be the same as the strategies in σ' under proposing partitions $Q \in \mathcal{Q}(\sigma')$. Observe that, because $V^{\sigma'} = V^{\sigma}$, the incentives to agree under $Q \in \mathcal{Q}(\sigma)$ are preserved, so for all $Q \in \mathcal{Q}(\sigma')$ we must have $Q \in \mathcal{Q}(\sigma)$. ■

Lemma 3 (Necessary Conditions for an Equilibrium). *Suppose that $(\mathcal{Q}_1, \mathcal{Q}_2)$ is a partition of $\mathcal{Q}$ and there exists an equilibrium σ such that $\mathcal{Q}(\sigma) = \mathcal{Q}_2$. Then $\delta V^{\sigma} > \frac{D_Q}{s_Q+1}$ for all $Q \in \mathcal{Q}_1$.*

Moreover, if σ is a simple equilibrium and $f(Q) > 0$ implies that $|\{k \in Q : k = D_Q\}| = 1$, then $\delta V^{\sigma} < \frac{D_Q}{s_Q+1}$ for all $Q \in \mathcal{Q}_2$.

Proof of lemma 3.

Because $\mathcal{Q}(\sigma) = \mathcal{Q}_2$, it must be the case that if $Q \in \mathcal{Q}_1$, then each proposer on $k \in Q$ pies prefers to make zero offers given that all other proposers in the proposing partition Q are making zero offers. The payoff to making the offers that are acceptable to the non-proposers is $k - s_Q \delta V^{\sigma}$, while the payoff to making zero offers is δV^{σ}. Thus it must be the case that $k - s_Q \delta V^{\sigma} < \delta V^{\sigma}$, which is equivalent to $\delta V^{\sigma} > \frac{k}{s_Q+1}$ for all $k \in Q$. Because $D_Q \in Q$, it must be the case that, in particular, $\delta V^{\sigma} > \frac{D_Q}{s_Q+1}$.

Suppose that σ is a simple equilibrium and $f(Q) > 0$ implies that $|\{q \in Q : q = D_Q\}| = 1$. This implies that for each proposing partition $Q \in \mathcal{Q}_2$ there is a unique agent who is a proposer on D_Q pies. Because $\mathcal{Q} \setminus \mathcal{Q}(\sigma) = \mathcal{Q}_1$ and σ is a simple equilibrium, it must be the case that if $Q \in \mathcal{Q}_2$, then the proposer on D_Q pies prefers to make an offer of δV^{σ} to each of the s_Q agents in a subset of non-proposers given that all other proposers in the proposing partition Q are making zero offers. The payoff to making these offers is $D_Q - s_Q\delta V^\sigma$, while the payoff to offering anything strictly less than δV^σ to any non-proposer is δV^σ. Thus it must be the case that $D_Q - s_Q\delta V^\sigma > \delta V^\sigma$, which is equivalent to $\delta V^\sigma < \frac{D_Q}{s_Q+1}$, as required. ■

Lemma 4. *Suppose that the distribution of proposal power is f and there exists an equilibrium σ. The value in an equilibrium σ with $\mathcal{Q}(\sigma) = \mathcal{P}$ is given by*

$$U(\mathcal{P}, f) = \frac{m}{n} \frac{\sum_{Q\in\mathcal{P}} f(Q)}{1 - \delta + \delta \sum_{Q\in\mathcal{P}} f(Q)}$$

Proof of lemma 4.

Consider an equilibrium in which there is agreement if and only if $Q \in \mathcal{P}$. Because under every proposing partition every agent has the same probability of being a proposer, if the agents use symmetric strategies in equilibrium, then their equilibrium values are the same.

Thus, because in a symmetric equilibrium the expected payoff in every state in which there is agreement must be the same for every agent and must equal the available surplus (which is m) divided by the number of agents (which is n), the value function V must satisfy

$$V = \frac{m}{n} \sum_{Q\in\mathcal{P}} f(Q) + \delta \left(1 - \sum_{Q\in\mathcal{P}} f(Q)\right) V$$

This is equivalent to

$$V = \frac{m}{n} \frac{\sum_{Q\in\mathcal{P}} f(Q)}{1 - \delta + \delta \sum_{Q\in\mathcal{P}} f(Q)}$$

as required. ■

Lemma 5 (Sufficient Conditions for an Equilibrium). *Suppose that $(\mathcal{Q}_1, \mathcal{Q}_2)$ is a partition of $\mathcal{Q}$, there exists a constant $\kappa > 0$ satisfying $\delta\kappa > \frac{D_Q}{s_Q+1}$ for $Q \in \mathcal{Q}_1$ and $\delta\kappa < \frac{D_Q}{s_Q+1}$ for $Q \in \mathcal{Q}_2$, and there exists a proposal power distribution f such that $\kappa = U(\mathcal{Q}_2, f)$. Then there exists a simple equilibrium σ such that $\mathcal{Q}(\sigma) = \mathcal{Q}_2$. Moreover, the value in the equilibrium σ is given by $U(\mathcal{Q}_2, f)$.*

Proof of lemma 5.

We will show that there exists a simple equilibrium σ such that $\mathcal{Q}(\sigma) = \mathcal{Q}_2$. By lemma 4, the value in an equilibrium σ is given by $U(\mathcal{Q}(\sigma), f)$. Because, by the hypothesis, there exists a proposal power distribution f such that $\kappa = U(\mathcal{Q}_2, f)$, κ is the value in the equilibrium σ.

We first show that if $Q \in \mathcal{Q}_1$, then each proposer on $k \in Q$ pies prefers to make zero offers given that all other proposers in the proposing partition Q are making zero offers. The payoff to making the offers that are acceptable to s_Q of the non-proposers is $k - s_Q\delta\kappa$, while the payoff to making zero offers is $\delta\kappa$. Thus we require that $k - s_Q\delta\kappa < \delta\kappa$, which is equivalent to $\delta\kappa > \frac{k}{s_Q+1}$ for all $k \in Q$. Because $D_Q \geq k$ for all $k \in Q$, the fact that $\delta\kappa > \frac{D_Q}{s_Q+1}$ for all $Q \in \mathcal{Q}_1$ implies that $\delta\kappa > \frac{k}{s_Q+1}$ for all $k \in Q$, $Q \in \mathcal{Q}_1$ is satisfied, as required.

Let N_Q denote the set of the non-proposers under Q. After the proposing partition Q is realized, the public randomization device generates a distribution that is uniform on $\{J \subseteq N_Q : |J| = s_Q\}$. Let $K = |\{k \in Q : k = D_Q\}|$. If the realization of the public randomization device is $J \subseteq N_Q$, then each proposer on D_Q pies makes an offer of $\frac{1}{K}\delta\kappa$ to each agent who is in J and makes zero offers to all other agents.

We next show that if $Q \in \mathcal{Q}_2$, then each of the K agents who is a proposer on D_Q pies weakly prefers to make an offer of $\frac{1}{K}\delta\kappa$ to each agent who is in J. The sum of offers each proposer on D_Q pies makes is $\frac{s_Q}{K}\delta\kappa$. The payoff to making the required offers is $D_Q - \frac{s_Q}{K}\delta\kappa$, while the payoff to deviating to offering a strictly smaller amount to any agent in J is $\delta\kappa$. Thus we require that $D_Q - \frac{s_Q}{K}\delta\kappa > \delta\kappa$, which is equivalent to $\delta\kappa < \frac{D_Q}{\frac{s_Q}{K}+1}$. Because $\delta\kappa < \frac{D_Q}{s_Q+1}$ for all $Q \in \mathcal{Q}_2$, we have $\delta\kappa < \frac{D_Q}{s_Q+1} \leq \frac{D_Q}{\frac{s_Q}{K}+1}$, as required.

Observe that the equilibrium σ is simple because only the proposers on the maximum number of pies D_Q make strictly positive offers. ■

I let $\Omega(f)$ denote the set of the agreement sets for the equilibria that exist given that the distribution of the proposal power is f.

Lemma 6 (Worst Equilibrium). *Given that the distribution of the proposal power is f, there exists a simple equilibrium σ such that $\mathcal{Q}(\sigma) = \mathcal{C}_f$. If for some equilibrium σ' we have that $\mathcal{Q}(\sigma') \in \Omega(f)$ and $\mathcal{Q}(\sigma') \neq \mathcal{Q}(\sigma)$, then $V^{\sigma'} > V^{\sigma}$.*

Proof of lemma 6.

Lemma 5 implies that if $\delta U(\mathcal{Q}(\sigma), f) > R_Q$ for all $Q \notin \mathcal{Q}(\sigma)$ and $\delta U(\mathcal{Q}(\sigma), f) < R_Q$ for all $Q \in \mathcal{Q}(\sigma)$, then there exists a simple equilibrium σ such that $\mathcal{Q}(\sigma) \in \Omega(f)$.

Recall that $\mathcal{C}_f = \{Q : R_Q \geq \delta V(R_Q)\}$, so that $U(\mathcal{C}_f, f) = V(R_{Q^*})$. Because $R_Q > R_{Q*}$ for all $Q \in \mathcal{C}_f \backslash Q^*$, this implies that $\delta U(\mathcal{C}_f, f) < R_Q$ for all $Q \in \mathcal{C}_f$. Because $R_{Q^*_-} < \delta V(R_{Q^*})$ by assumption and $R_Q < R_{Q^*_-}$ for all $Q \notin \mathcal{C}_f \cup Q^*_-$, we have $\delta U(\mathcal{C}_f, f) > R_Q$ for all $Q \notin \mathcal{C}_f$. Therefore, there exists a simple equilibrium σ such that $\mathcal{Q}(\sigma) = \mathcal{C}_f$, as required.

Suppose that there exists an equilibrium σ' such that $\mathcal{Q}(\sigma') \in \Omega(f)$ and $\mathcal{Q}(\sigma') \neq \mathcal{Q}(\sigma)$.

If $\mathcal{Q}(\sigma) \subset \mathcal{Q}(\sigma')$, then we are done. Therefore, suppose that there exists $Q' \in \mathcal{Q}(\sigma)$ such that $Q' \notin \mathcal{Q}(\sigma')$. By lemma 3, a necessary condition to have $\mathcal{Q}(\sigma') \in \Omega(f)$ is that $\delta V^{\sigma'} > R_Q$ for all $Q \in \mathcal{Q}$ such that $Q \notin \mathcal{Q}(\sigma')$. In particular, because $Q' \notin \mathcal{Q}(\sigma')$, we must have

$$\delta V^{\sigma'} > R_{Q'} \tag{1}$$

Recall that $Q^* = \min\{Q : Q \in \mathcal{C}_f\}$, where the minimum is taken with respect to the order induced by R_Q, and that $V^{\sigma} = U(\mathcal{C}_f, f)$. Then

$$R_{Q^*} \geq \delta V^{\sigma} \tag{2}$$

Because $Q' \in \mathcal{Q}(\sigma)$, we have

$$R_{Q'} \geq R_{Q^*} \tag{3}$$

Thus $\delta V^{\sigma'} > R_{Q'} \geq R_{Q^*} \geq \delta V^{\sigma}$, where the first inequality follows from (1), the second inequality follows from (3) and the third inequality follows from (2). Therefore, $V^{\sigma'} > V^{\sigma}$, as required. ■

Proof of Theorem 1.

The proof of Part 1 of the Theorem follows from lemma 7. The claim that in the worst equilibrium, the agreement set is $\mathcal{C}_f = \{Q \in \mathcal{Q} : R_Q \geq \delta V(R_Q)\}$ follows from lemma 6.

We now claim that there exists an equilibrium with delayed agreement if and only if $\mathcal{C}_f \neq \mathcal{Q}$. If $\mathcal{C}_f \neq \mathcal{Q}$, then lemma 6 implies that there exists an equilibrium with delayed agreement. If there exists an equilibrium with delayed agreement, then the payoff in this equilibrium is strictly less than $\frac{m}{n}$, which implies that $\mathcal{C}_f \neq \mathcal{Q}$.

Finally, we show that $\mathcal{C}_f \neq \mathcal{Q}$ if and only if $\min_{Q \in \mathcal{Q}} R_Q < \delta \frac{m}{n}$. This follows from the fact that $\mathcal{C}_f = \{Q : R_Q \geq \delta V(R_Q)\}$, V is decreasing and $V(\min_{Q \in \mathcal{Q}} R_Q) = \frac{m}{n}$. ■

Definition 2. Given $\mathcal{P} \subset \mathcal{Q}$, we say that $L(\mathcal{P})$ is the *disagreement partition* if

$$L(\mathcal{P}) = \max \mathcal{Q} \setminus \mathcal{P}$$

where the maximum is taken with respect to the order on proposing partitions induced by R_Q.

I let $B_Q = 1 - \frac{1-\delta}{\delta\left(\frac{m}{n}\frac{s_Q+1}{D_Q}-1\right)}$ for Q such that $R_Q < \frac{m}{n}$ and $B_Q = -\infty$ for Q such that $R_Q \geq \frac{m}{n}$.

Lemma 7. *If* $\sum_{Q\in\mathcal{P}} f(Q) > 1 - B_{L(\mathcal{P})}$ *or* $\mathcal{P} = \mathcal{Q}$*, then* $\mathcal{P} \in \Omega(f)$*.*

Proof of lemma 7.

We will show that there exists an equilibrium σ such that $\mathcal{Q}(\sigma) = \mathcal{P}$. Suppose first that $\mathcal{P} \neq \mathcal{Q}$. We require that there is disagreement under the proposing partitions not in $\mathcal{P}$. We conjecture equilibrium strategies such that under the proposing partitions not in $\mathcal{P}$, all proposers make zero offers. For this, it is sufficient that, for each $Q \in \mathcal{Q} \setminus \mathcal{P}$, provided that all other proposers are making zero offers, a proposer on D_Q pies prefers not to make the offers leading to an agreement. In order to obtain agreement, a proposer on D_Q pies would need to make offers yielding a payoff no greater than $D_Q - s_Q\delta V^\sigma$ to the proposer, while the payoff to disagreeing is δV^σ. Thus it is sufficient to have $D_Q - s_Q\delta V^\sigma < \delta V^\sigma$ for all $Q \in \mathcal{Q} \setminus \mathcal{P}$, which is equivalent to $\delta V^\sigma > \frac{D_Q}{s_Q+1}$ for all $Q \in \mathcal{Q} \setminus \mathcal{P}$.

Recall that $L(\mathcal{P}) = \max\{Q : Q \in \mathcal{Q} \setminus \mathcal{P}\}$, where the maximum is taken with respect to the order induced by R_Q. Because, by lemma 4, we have $V^\sigma = U(\mathcal{P}, f)$, $\sum_{Q\in\mathcal{P}} f(Q) > 1 - B_{L(\mathcal{P})}$ is equivalent to $\delta V^\sigma > \frac{D_{L(\mathcal{P})}}{s_{L(\mathcal{P})}+1}$. Because $R_{L(\mathcal{P})} \geq R_Q$ for all $Q \notin \mathcal{P}$, this implies that $\delta V^\sigma > \frac{D_Q}{s_Q+1}$ for all $Q \notin \mathcal{P}$.

Suppose next that $\mathcal{P} = \mathcal{Q}$. Then $\mathcal{Q} \setminus \mathcal{P} = \varnothing$, and it is sufficient to show that there is agreement under the proposing partitions in $\mathcal{P}$.

Given $Q \in \mathcal{Q}$ and $k \in Q$, define

$$\beta_k(Q) = \frac{kn - m}{s_Q m}$$

We require that there is agreement under the proposing partitions in $\mathcal{P}$. For each $Q \in \mathcal{P}$, we conjecture the following strategies. The public randomization device generates a distribution that is uniform on $\{J \subseteq N_Q : |J| = s_Q\}$, where N_Q denotes the set of the non-proposers under the proposing partition Q. If the realization of the public randomization device is $J \subseteq N_Q$, then each proposer controlling k pies in Q offers $\beta_k(Q)\delta V^\sigma$ to each agent in J. Observe that $\beta_k(Q) > 0$ and

$$\sum_{k\in Q} \beta_k(Q) = \frac{\sum_{k\in Q} kn}{s_Q m} - \frac{|Q|}{s_Q} = \frac{mn}{s_Q m} - \frac{n - s_Q}{s_Q} = 1$$

Then each non-proposer receives the sum of offers equal to $\sum_{k\in Q} \beta_k(Q)\delta V^\sigma = \delta V^\sigma$ and accepts these offers.

The payoff to making these offers is $k - s_Q\beta_k(Q)\delta V^\sigma$, while the payoff to deviating to offers that are any lower is δV^σ. Thus we require that $k - s_Q\beta_k(Q)\delta V^\sigma > \delta V^\sigma$, which is equivalent to $\delta V^{\sigma} < \frac{k}{s_Q \beta_k(Q)+1}$. Because $\delta V^{\sigma} < \frac{m}{n}$, it is sufficient to show that

$$\frac{m}{n} \leq \frac{k}{s_Q \beta_k(Q) + 1}$$

The definition of $\beta_k(Q)$ ensures that this constraint holds with equality. ■

Lemma 8. *If $\tilde{\mathcal{P}} \in \Omega(f)$, then for all $\mathcal{P} \subseteq \mathcal{Q}$ such that $\tilde{\mathcal{P}} \subset \mathcal{P}$, we have $\mathcal{P} \in \Omega(f)$.*

Proof of lemma 8.

Suppose first that $\mathcal{P} = \mathcal{Q}$. Then the result follows from lemma 7.

Suppose next that $\mathcal{P} \subset \mathcal{Q}$. Then $\mathcal{Q} \setminus \mathcal{P} \neq \varnothing$, so $L(\mathcal{P})$ is well-defined.

Because $\tilde{\mathcal{P}} \in \Omega(f)$, there exists an equilibrium σ such that $\mathcal{Q}(\sigma) = \tilde{\mathcal{P}}$. Then there is disagreement in equilibrium σ under proposing partitions in $\mathcal{Q} \setminus \tilde{\mathcal{P}}$. Because $\tilde{\mathcal{P}} \subset \mathcal{P}$, we have $\mathcal{Q} \setminus \mathcal{P} \subset \mathcal{Q} \setminus \tilde{\mathcal{P}}$. This implies that there is disagreement in equilibrium σ under proposing partitions in $\mathcal{Q} \setminus \mathcal{P}$.

Because there is disagreement in equilibrium σ under proposing partitions in $\mathcal{Q} \setminus \mathcal{P}$, lemma 3 implies that $\delta V^{\sigma} > R_Q$ for all $Q \in \mathcal{Q} \setminus \mathcal{P}$. Because $V^{\sigma} = U(\mathcal{Q}(\sigma), f)$, this is equivalent to

$$\sum_{Q \in \tilde{\mathcal{P}}} f(Q) > 1 - B_{L(\tilde{\mathcal{P}})} \tag{4}$$

Because $\tilde{\mathcal{P}} \subset \mathcal{P}$, we have that $L(\mathcal{P}) \leq L\left(\tilde{\mathcal{P}}\right)$. Because $Q \mapsto B_Q$ is decreasing, this implies that

$$1 - B_{L(\mathcal{P})} \leq 1 - B_{L(\tilde{\mathcal{P}})} \tag{5}$$

Because $\tilde{\mathcal{P}} \subset \mathcal{P}$, we have

$$\sum_{Q \in \mathcal{P}} f(Q) \geq \sum_{Q \in \tilde{\mathcal{P}}} f(Q) \tag{6}$$

Then

$$\sum_{Q \in \mathcal{P}} f(Q) \geq \sum_{Q \in \tilde{\mathcal{P}}} f(Q) > 1 - B_{L(\tilde{\mathcal{P}})} \geq 1 - B_{L(\mathcal{P})}$$

where the first inequality follows from (6), the second inequality follows from (4) and the third inequality follows from (5).

Thus $\sum_{Q \in \mathcal{P}} f(Q) > 1 - B_{L(\mathcal{P})}$ holds. Then the result follows from lemma 7. ■

Proof of Proposition 1.

Proof of Part 1 of the Proposition.

We will prove the following statement: for $\mathcal{P} \in \Omega(f)$, if $\sum_{Q\in\mathcal{P}} \tilde{f}(Q) \geq \sum_{Q\in\mathcal{P}} f(Q)$, then $\mathcal{P} \in \Omega\left(\tilde{f}\right)$.

Consider $\mathcal{P} \subset \mathcal{Q}$. We will show that if $\mathcal{P} \subset \mathcal{Q}$, then $\mathcal{P} \in \Omega(f)$ if and only if $\sum_{Q\in\mathcal{P}} f(Q) > 1 - B_{L(\mathcal{P})}$. Part 1 of the Proposition will then follow from this.

Suppose that $\mathcal{P} \in \Omega(f)$. By lemma 3, a necessary condition to have $\mathcal{P} \in \Omega(f)$ is that $\delta U(\mathcal{P}, f) > R_Q$ for all $Q \notin \mathcal{P}$. In particular, we must have $\delta U(\mathcal{P}, f) > \frac{D_{L(\mathcal{P})}}{s_{L(\mathcal{P})}+1}$. This implies that $\sum_{Q\in\mathcal{P}} f(Q) > 1 - B_{L(\mathcal{P})}$, as required. Conversely, suppose that $\sum_{Q\in\mathcal{Q}(\sigma)} f(Q) > 1 - B_{L(\mathcal{Q}(\sigma))}$. Then the result follows from lemma 7.

Proof of Part 2 of the Proposition.

We will prove the following statement: $\mathcal{G} \subseteq \mathcal{P}$ for $\mathcal{P} \in \Omega(f)$ and $\{m\} \in \mathcal{G}$.

Lemma 3 implies that if $\delta V(R_Q) \leq R_Q$, then agents must agree at Q. Note that the fact that $V(R_Q) \leq \frac{m}{n}$ for all Q implies that $\delta V(R_Q) \leq \delta\frac{m}{n}$. Then for Q such that $\delta\frac{m}{n} \leq R_Q$, we have $\delta V(R_Q) \leq R_Q$. This implies that at all Q such that $\delta\frac{m}{n} \leq R_Q$, agents must agree in any equilibrium.

Proof of Part 3 of the Proposition.

We will prove the following statement: $\{\mathcal{P} \subseteq \mathcal{Q} : \mathcal{C}_f \subseteq \mathcal{P}\} \subseteq \Omega(f)$.

Lemma 6 implies that $\mathcal{C}_f \in \Omega(f)$. The result then follows from lemma 8.

Proof of Part 4 of the Proposition.

We will prove the following statement: if $\sum_{Q\notin\mathcal{C}_f} f(Q) < f(Q')$ for all $Q' \in \mathcal{C}_f \setminus \mathcal{G}$, then $\{\mathcal{P} \subseteq \mathcal{Q} : \mathcal{C}_f \subseteq \mathcal{P}\} = \Omega(f)$.

Suppose that $\sum_{Q\notin\mathcal{C}_f} f(Q) < f(Q')$ for all $Q' \in \mathcal{C}_f \setminus \mathcal{G}$. Part 3 of the Proposition shows that $\{\mathcal{P} \subseteq \mathcal{Q} : \mathcal{C}_f \subseteq \mathcal{P}\} \subseteq \Omega(f)$. Given this, it is sufficient to show that $\Omega(f) \subseteq \{\mathcal{P} \subseteq \mathcal{Q} : \mathcal{C}_f \subseteq \mathcal{P}\}$.

Then we would like to show that there does not exist $\mathcal{P} \in \Omega(f)$ such that $Q' \notin \mathcal{P}$ for some $Q' \in \mathcal{C}_f$. Thus we fix $\mathcal{P}$ such that there exists Q' satisfying $Q' \notin \mathcal{P}$ and $Q' \in \mathcal{C}_f$. We will show that $\mathcal{P} \notin \Omega(f)$.

Because $Q' \notin \mathcal{P}$, we have $L(\mathcal{P}) \geq Q'$. Because $Q \mapsto B_Q$ is decreasing, this implies that

$$1 - B_{L(\mathcal{P})} \geq 1 - B_{Q'} \tag{7}$$

Observe that $\sum_{Q\in\mathcal{P}} f(Q) \leq \sum_{Q\neq Q'} f(Q) = 1 - f(Q')$. Because $\sum_{Q\notin\mathcal{C}_f} f(Q) < f(Q')$

by the hypothesis, we have $1 - f(Q') < 1 - \sum_{Q \notin \mathcal{C}_f} f(Q) = \sum_{Q \in \mathcal{C}_f} f(Q)$. This implies that

$$\sum_{Q \in \mathcal{P}} f(Q) < \sum_{Q \in \mathcal{C}_f} f(Q) \tag{8}$$

Recall that $\delta U(\mathcal{C}_f, f) \leq \min_{Q \in \mathcal{C}_f} R_Q$ and $Q^* = \min \mathcal{C}_f$, where the minimum is taken with respect to the order induced by R_Q. Then $\delta U(\mathcal{C}_f, f) \leq \min_{Q \in \mathcal{C}_f} R_Q$ implies that

$$\sum_{Q \in \mathcal{C}_f} f(Q) \leq 1 - B_{Q^*} \tag{9}$$

Therefore, $\sum_{Q \in \mathcal{P}} f(Q) < \sum_{Q \in \mathcal{C}_f} f(Q) \leq 1 - B_{Q^*} \leq 1 - B_{Q'} \leq 1 - B_{L(\mathcal{P})}$, where the first inequality follows from (8), the second inequality follows from (9), the third inequality follows from the fact that $Q' \geq Q^*$ since $Q' \in \mathcal{C}_f$ and from the fact that $Q \mapsto B_Q$ is decreasing, and the fourth inequality follows from (7).

Thus we have $\sum_{Q \in \mathcal{P}} f(Q) < 1 - B_{L(\mathcal{P})}$. Then part 1 of the Proposition implies that $\mathcal{P} \notin \Omega(f)$, as required. ■

Proof of Proposition 2.

Suppose first that $\min_{Q \in \mathcal{Q}} R_Q > \delta \frac{m}{n}$. If one period remains in the game, then there is immediate agreement yielding the expected payoff $\frac{m}{n}$ to each agent. If two periods remain, then a proposer on D_Q pies strictly prefers to make offers leading to agreement when other proposers make zero offers if $D_Q - s_Q \delta \frac{m}{n} > \delta \frac{m}{n}$, which is equivalent to $R_Q > \delta \frac{m}{n}$. Since $\min_{Q \in \mathcal{Q}} R_Q > \delta \frac{m}{n}$, the inequality is satisfied for all Q, which implies that there is agreement under all $Q \in \mathcal{Q}$. By induction, when T periods remain, there is agreement under all $Q \in \mathcal{Q}$.

Suppose next that $\min_{Q \in \mathcal{Q}} R_Q \leq \delta \frac{m}{n}$. Fix an equilibrium σ of the infinite-horizon game and let $\mathcal{P} = \mathcal{Q}(\sigma)$. Let $V_t\left(\sigma^T\right)$ denote the continuation value given the equilibrium σ^T when t periods have passed in a T-period game. For $t \leq T$, let $\mathcal{K}_{t-1} = \left\{Q \in \mathcal{Q} : R_Q \geq \delta V_t\left(\sigma^T\right)\right\}$.

Set $\mathcal{Q}_{T-1}\left(\sigma^T\right) = \mathcal{Q}$ and $\mathcal{Q}_t\left(\sigma^T\right) = \mathcal{P}$ for all $t < T - 1$. Let $\tilde{V}$ denote the value in the equilibrium σ in the infinite-horizon game. Note that $V_{T-1} = \frac{m}{n}$ and $V_{t-1}\left(\sigma^T\right) = \sum_{Q \in \mathcal{P}} f(Q) \frac{m}{n} + \left(1 - \sum_{Q \in \mathcal{P}} f(Q)\right) \delta V_t\left(\sigma^T\right)$ for $t < T$. This implies that $V_t\left(\sigma^T\right) \in \left(\tilde{V}, \frac{m}{n}\right]$ for all $t < T$.

Consider Q' such that $R_{Q'} = \max_{Q \in \mathcal{Q} \setminus \mathcal{P}} R_Q$. Because $Q' \notin \mathcal{P}$, we have $R_{Q'} \leq \delta \tilde{V}$. Then, because $V_t\left(\sigma^T\right) \in \left(\tilde{V}, \frac{m}{n}\right]$, we have $R_{Q'} < \delta V_t\left(\sigma^T\right)$ for all $t < T$. This implies that $Q' \notin \mathcal{K}_t$ for all $t < T - 1$. A proof similar to the proofs of lemma 5 and Proposition 1 can be used to show that we can sustain agreement at $Q \in \mathcal{Q}_t\left(\sigma^T\right)$ by conjecturing that all proposers contribute appropriate amounts, and that, since $R_{Q'} = \max_{Q \in \mathcal{Q} \setminus \mathcal{P}} R_Q$ and $Q' \notin \mathcal{K}_t$ for $t < T-1$, disagreement is sustainable at all $Q \in \mathcal{Q} \setminus \mathcal{P}$ for $t < T-1$. This implies that the equilibrium σ^T exists. ■

Proof of Proposition 3.

The first part of the Proposition follows from lemma 7. Lemma 3 implies that if $|\{k \in Q : k = D_Q\}| = 1$ whenever $f(Q) > 0$, then if σ is a simple equilibrium, $\delta U(\mathcal{Q}(\sigma), f) > R_Q$ for all $Q \in \mathcal{Q} \setminus \mathcal{Q}(\sigma)$ and $\delta U(\mathcal{Q}(\sigma), f) < R_Q$ for all $Q \in \mathcal{Q}(\sigma)$. By definition of $\mathcal{C}_f$, this is equivalent to $\mathcal{Q}(\sigma) = \mathcal{C}_f$. Therefore, the agreement set $\mathcal{Q}(\sigma)$ for the simple equilibrium is unique, which implies that the simple equilibrium is unique. ■

Proof of Proposition 4.

We will prove the following statement: if σ_q is a worst equilibrium under the voting rule q and $q_1 \geq q_2$, then $\sum_{Q \in \mathcal{Q}(\sigma_{q_1})} f(Q) \leq \sum_{Q \in \mathcal{Q}(\sigma_{q_2})} f(Q)$ and $U(\mathcal{Q}(\sigma_{q_1}), f) \leq U(\mathcal{Q}(\sigma_{q_2}), f)$.

We make the dependence of the number of supporters s_Q on q explicit by writing s_Q^q. Recall that, by assumption, $q > m$ for all q, which implies that $s_Q^q = q - |Q| > 0$ for all Q and q. Define $J_Q^{q,p} = \delta \frac{m}{n} \frac{p}{1-\delta+\delta p} - \frac{D_Q}{s_Q^q+1}$. Observe that if for all $Q \in \mathcal{Q}$ and q_1, q_2 we have $s_Q^{q_1} > 0$ and $s_Q^{q_2} > 0$, then whenever $q_1 > q_2$, we have $s_Q^{q_1} > s_Q^{q_2}$. This implies that $\partial J_Q^{q,p}/\partial q > 0$. Observe also that $p \mapsto \delta \frac{m}{n} \frac{p}{1-\delta+\delta p}$ is strictly increasing. This implies that $\partial J_Q^{q,p}/\partial p < 0$. Letting $p(q)$ be implicitly defined by $J_Q^{q,p(q)} = 0$, observe that then we have $\frac{\partial p(q)}{\partial q} = -\frac{\partial J_Q^{q,p}/\partial q}{\partial J_Q^{q,p}/\partial p} < 0$.

By lemma 6, the agreement set in the worst equilibrium is given by $\mathcal{C}_f^q = \left\{Q \in \mathcal{Q} : \delta \frac{m}{n} \frac{\sum_{Q \in \mathcal{Q}(\sigma_q)} f(Q)}{1-\delta+\delta \sum_{Q \in \mathcal{Q}(\sigma_q)} f(Q)} - \frac{D_Q}{s_Q^q+1} \leq 0\right\}$, where I have made the dependence of the worst equilibrium agreement set on q explicit. Therefore, $\mathcal{C}_f^q = \left\{Q \in \mathcal{Q} : J_Q^{q,p} \leq 0\right\}$ for $p = \sum_{Q \in \mathcal{Q}(\sigma_q)} f(Q)$. Then the fact that $\frac{\partial p(q)}{\partial q} < 0$ implies that $\sum_{Q \in \mathcal{Q}(\sigma_{q_1})} f(Q) \leq \sum_{Q \in \mathcal{Q}(\sigma_{q_2})} f(Q)$, as required. ■

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